

TTIC 31260 Algorithmic Game Theory

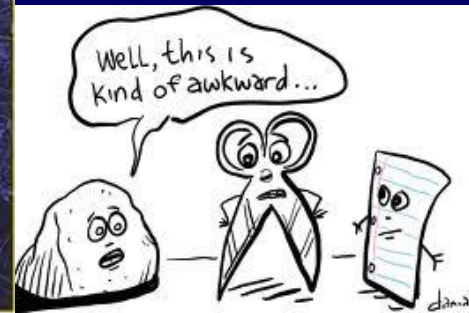
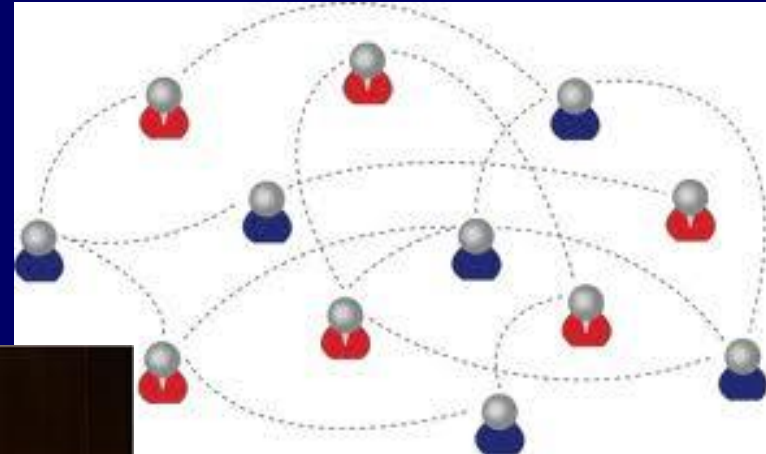
Winter 2026

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TTIC

Overall

Theory and algorithms for systems of interacting agents, each with their own interests in mind.



Topics (subject to change)

- Basics of game theory, equilibria
- Connections between learning and equilibria
- Quality of equilibria: price of anarchy
- Social choice: voting, manipulation
- Mechanism design: designing rules of the game to achieve desired outcome, auctions.
- Fair division
- Matching markets

Admin

Free online book: “Algorithmic game theory”

Course requirements:

- 4 homeworks [60%]
- Course project [20%]
 - Could be reading and explaining a recent paper related to class topics
 - Could be theoretically investigating a question related to class topics
 - Could be conducting an experimental investigation
- Class participation including helping to grade one homework [20%]

01/05/26

A Basic Introduction to Game Theory

[Readings: Ch. 1.1-1.3 of AGT book]

Game theory

- Field developed by economists to study social & economic interactions.
 - Wanted to understand why people behave the way they do in different economic situations. Effects of incentives. Rational explanation of behavior.
- “Game” = interaction between parties with their own interests. Could be called “interaction theory”.
- Big in CS for understanding large systems:
 - Internet routing, social networks, e-commerce
 - Problems like spam etc.

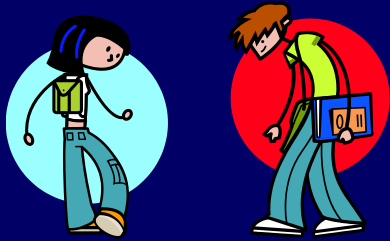
Setting

- Have a collection of participants, or *players*.
- Each has a set of choices, or *strategies* for how to play/behave.
- Combined behavior results in *payoffs* (satisfaction level) for each player.

Most examples today will involve just 2 players
(which will make them easier to picture, as will
become clear in a moment...)

Example: walking on the sidewalk

- What side of ~~sidewalk~~ ^{street to drive on} should I walk on?
- Two options for you (left or right). Same for person walking towards you.
- Can describe payoffs in matrix:



you

Left

Right

Left Right

(1,1)	(-1,-1)
(-1,-1)	(1,1)

person
walking
towards you

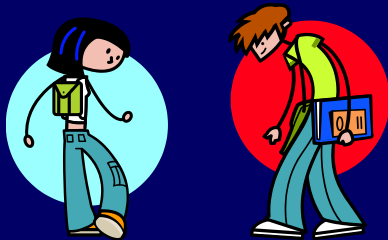
Your payoff for RR

His payoff for RR

Key notion

Could be randomized

- "Nash Equilibrium": pair of strategies such that each player is playing a best-response to the other. Neither has an incentive to change.



you

Left

Right

Left Right

(1,1)	(-1,-1)
(-1,-1)	(1,1)

person
walking
towards you

Your payoff for RR

His payoff for RR

Example: prisoner's dilemma

- Consider two companies deciding whether to install pollution controls.
- Imagine pollution controls cost \$4 but improve everyone's environment by \$3

	control	don't control
control	(2,2)	(-1,3)
don't control	(3,-1)	(0,0)

For both, defecting is dominant strategy

What do equilibria look like here?

Example: prisoner's dilemma

- Consider two companies deciding whether to install pollution controls.
- Imagine pollution controls cost \$4 but improve everyone's environment by \$3

	control	don't control
control	(2,2)	(-1,3)
don't control	(3,-1)	(0,0)

For both, defecting is dominant strategy

Need to add extra incentives to get good overall behavior.

Example: matching pennies / penalty shot

- Shooter can choose to shoot left or shoot right.
- Goalie can choose to dive left or dive right.
- If goalie guesses correctly, (s)he saves the day. If not, it's a **gooooooooooalll!** Vice-versa for shooter.

		goalie	
		Left	Right
shooter	Left	(0,0)	(1,-1)
	Right	(1,-1)	(0,0)

GOAALLL!!!

No deterministic equilibrium

Example: matching pennies / penalty shot

- Shooter can choose to shoot left or shoot right.
- Goalie can choose to dive left or dive right.
- If goalie guesses correctly, (s)he saves the day. If not, it's a **gooooooooooalll!** Vice-versa for shooter.

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		Left	Right
shooter	Left	(0,0)	(1,-1)
	Right	(1,-1)	(0,0)

GOAALLL!!!

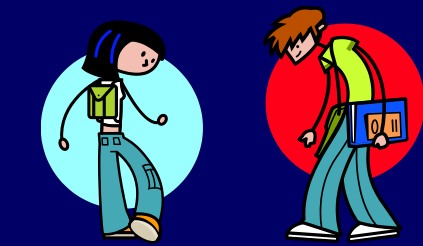
Each playing 50/50 is a Nash equilibrium

Nash (1950)

- Proved that if you allow *randomized* (mixed) strategies then every game has at least one equilibrium.
- I.e., a *pair of (randomized) strategies* that is *stable* in the sense that each is a best response to the other in terms of expected payoff.
- For this, and its implications, Nash received the Nobel prize.

Game theory terminology

- Rows and columns called pure strategies.
- Randomized algs called mixed strategies.



you

Left

Right

Left Right

(1,1)	(-1,-1)
(-1,-1)	(1,1)

person
walking
towards
you

Your payoff for RR

His payoff for RR

Game theory terminology

- Rows and columns called pure strategies.
- Randomized algs called mixed strategies.
- Often describe in terms of 2 matrices R, C .

R

1	-1
-1	1

C

1	-1
-1	1

(p, q) is Nash equilib if $p^T R q \geq e_i^T R q \forall i$ and $p^T C q \geq p^T C e_j \forall j$.

Basic facts

- (p,q) is NashEq if $p^T R q \geq e_i^T R q \forall i, p^T C q \geq p^T C e_j \forall j$.
- $\Rightarrow$ for all i s.t. $p_i > 0$ we have $e_i^T R q = \max_{i'} e_{i'}^T R q$
- $\Rightarrow$ for all j s.t. $q_j > 0$ we have $p^T C e_j = \max_{j'} p^T C e_{j'}$

R

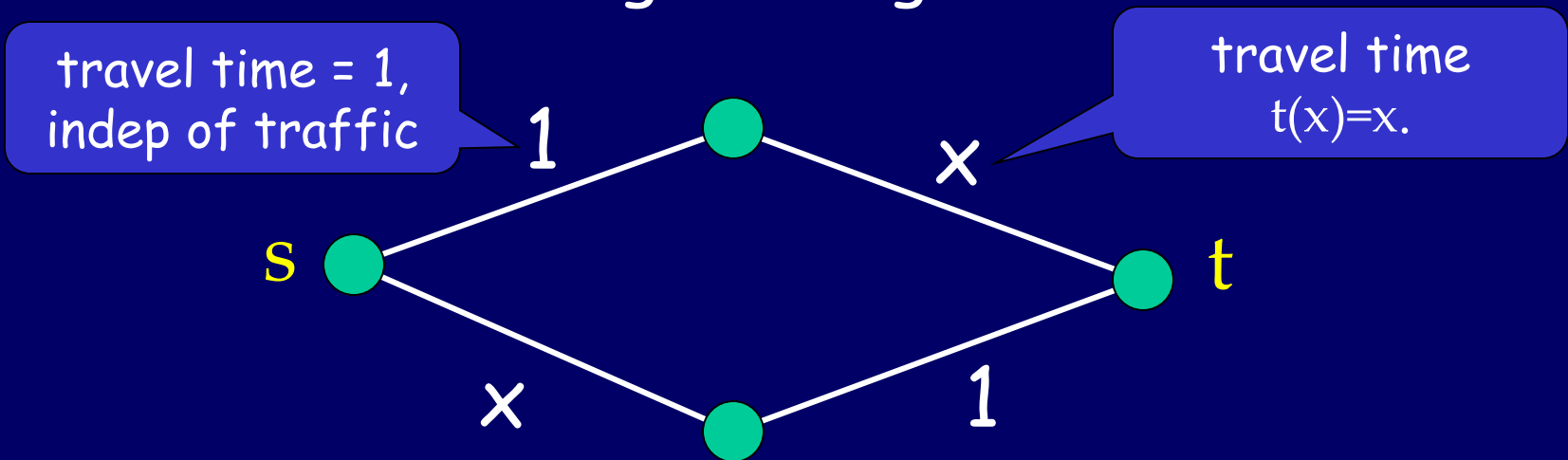
1	-1
-1	1

C

1	-1
-1	1

NE can do strange things

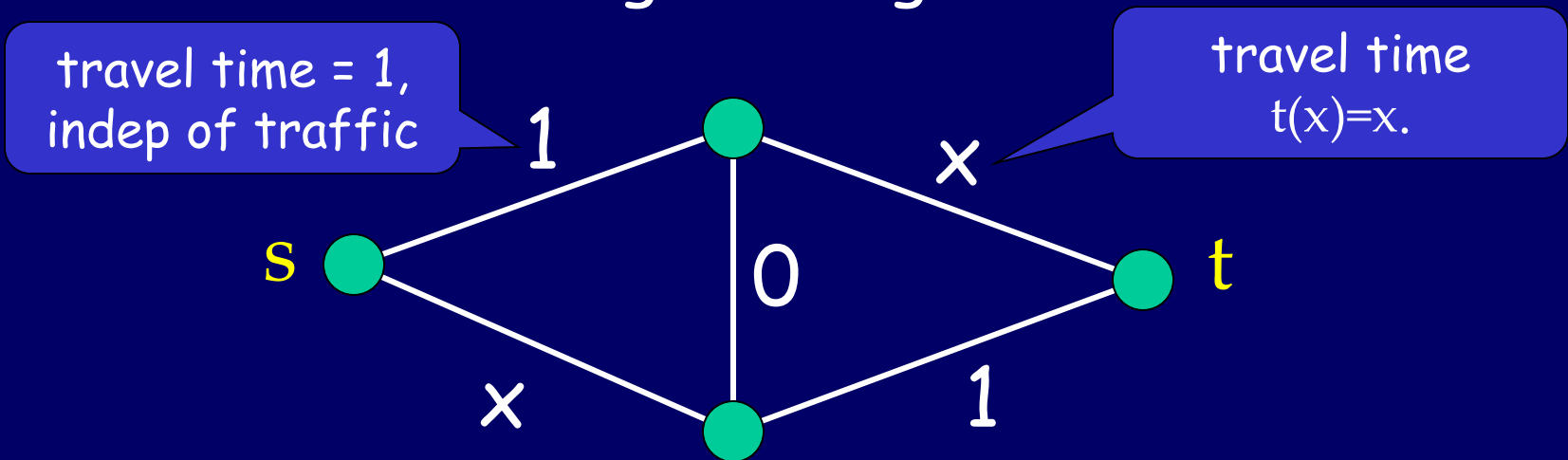
- Braess paradox:
 - Road network, traffic going from **s** to **t**.
 - travel time as function of fraction **x** of traffic on a given edge.



Fine. NE is 50/50. Travel time = 1.5

NE can do strange things

- Braess paradox:
 - Road network, traffic going from **s** to **t**.
 - travel time as function of fraction **x** of traffic on a given edge.



Add new superhighway. NE: everyone uses zig-zag path. Travel time = 2.

2-Player Zero-Sum games

- Zero-sum games are the special case of purely-competitive 2-player games.
 - Recall: an entry (x,y) means: x = payoff to row player, y = payoff to column player. "Zero sum" means that $y = -x$.
- E.g., matching pennies / penalty shot:

		Left	Right	goalie
shooter	Left	(0,0)	(1,-1)	
	Right	(1,-1)	(0,0)	GOAALLL!!!

Minimax-optimal strategies

- Minimax optimal strategy is a (randomized) strategy that has the best guarantee on its expected gain, over choices of the opponent.
[maximizes the minimum]
- I.e., the thing to play if your opponent knows you well.

		Left	Right	goalie
shooter	Left	(0,0)	(1,-1)	GOAALLL!!!
	Right	(1,-1)	(0,0)	

Minimax optimal for both players is 50/50. Gives expected gain of $\frac{1}{2}$ for shooter, $-\frac{1}{2}$ for goalie. Any other is worse.

Minimax-optimal strategies

- How about penalty shot with goalie who's weaker on the left?

Say shooter uses $(p, 1-p)$.

- If goalie dives left, shooter gets $p/2 + 1-p = 1 - p/2$.
- If goalie dives right, shooter gets p .
- Maximize minimum by setting equal.
- Gives $p = 2/3$.

		goalie	
		Left	Right
shooter	Left	$(\frac{1}{2}, -\frac{1}{2})$	$(1, -1)$
	Right	$(1, -1)$	$(0, 0)$

GOAALLL!!!

50/50

Minimax-optimal strategies

- How about penalty shot with goalie who's weaker on the left?

Minimax optimal for shooter is $(2/3, 1/3)$.

Guarantees expected gain at least $2/3$.

Minimax optimal for goalie is also $(2/3, 1/3)$.

Guarantees expected loss at most $2/3$.

		goalie	
		Left	Right
shooter	Left	$(\frac{1}{2}, -\frac{1}{2})$	$(1, -1)$
	Right	$(1, -1)$	$(0, 0)$

GOAALLL!!!

50/50

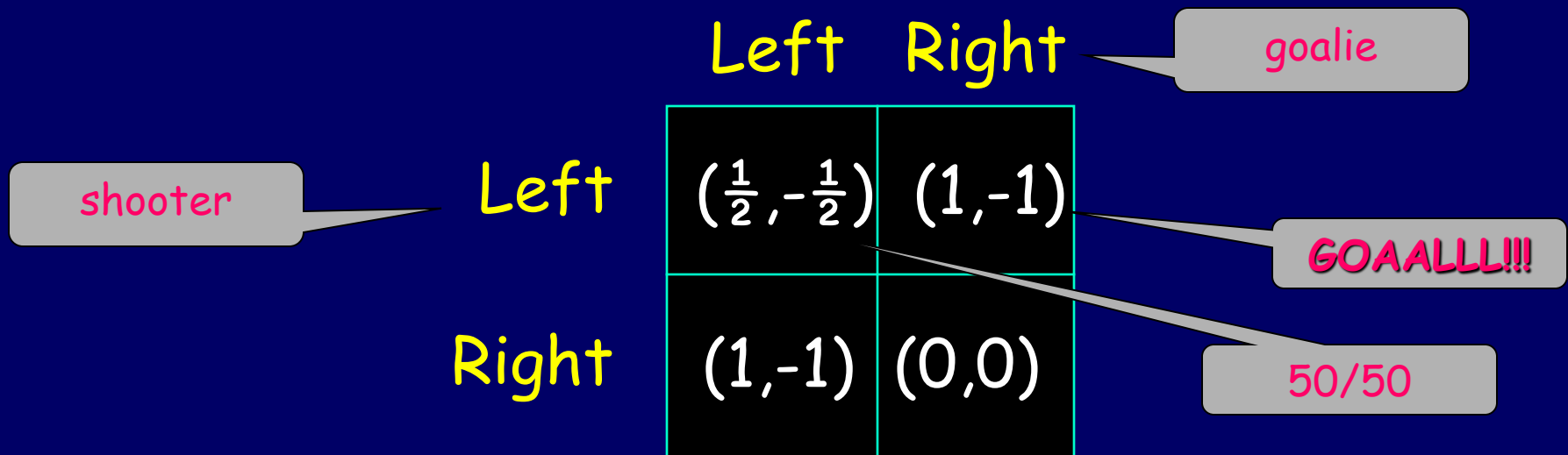
Minimax-optimal strategies

- Can solve for minimax optimal strategy using Linear Programming:

Variables p, v .

Maximize v subject to:

- $p \cdot R_j \geq v$, for all j . (R_j is j th column of R)
- p is legal prob dist ($p_i \geq 0, \sum_i p_i = 1$).



Minimax Theorem (von Neumann 1928)

- Every 2-player zero-sum game has a unique value V .
- Minimax optimal strategy for Row guarantees Row's expected gain at least V .
- Minimax optimal strategy for Col guarantees Col's expected loss at most V .

Counterintuitive: Means it doesn't hurt to publish your strategy if both players are optimal. (Borel had proved for symmetric 5×5 but thought was false for larger games)

Nash $\Rightarrow$ Minimax

- Nash's theorem actually gives minimax thm as a corollary.
 - Pick some NE and let V = value to row player in that equilibrium.
 - Since it's a NE, neither player can do better even knowing the (randomized) strategy their opponent is playing.
 - So, they're each playing minimax optimal.

Nash $\Rightarrow$ Minimax

- On the other hand, for minimax, also have very constructive, algorithmic arguments:
 - Can solve for minimax optimum using linear programming in time $\text{poly}(n)$ ($n = \text{size of game}$)
 - Have adaptive procedures that in repeated play guarantee to approach/beat best fixed strategy in hindsight
- But for Nash, no efficient procedures to find: NP-hard to find equilib with special properties, PPAD-hard just to find one.

Can use notion of minimax
optimality to explain bluffing
in poker

Simplified Poker (Kuhn 1950)

- Two players **A** and **B**.
- Deck of 3 cards: **1,2,3**.
- Players ante \$1.
- Each player gets one card.
- **A** goes first. Can bet \$1 or pass.
 - If **A** bets, **B** can call or fold.
 - If **A** passes, **B** can bet \$1 or pass.
 - If **B** bets, **A** can call or fold.
- High card wins (if no folding). Max pot \$2.

- Two players **A** and **B**. 3 cards: **1,2,3**.
- Players ante \$1. Each player gets one card.
- **A** goes first. Can bet \$1 or pass.
 - If **A** bets, **B** can call or fold.
 - If **A** passes, **B** can bet \$1 or pass.
 - If **B** bets, **A** can call or fold.

Writing as a Matrix Game

- For a given card, **A** can decide to
 - Pass but fold if **B** bets. [PassFold]
 - Pass but call if **B** bets. [PassCall]
 - Bet. [Bet]
- Similar set of choices for **B**.

Can look at all strategies as a big matrix...

	[FP,FP,CB]	[FP,CP,CB]	[FB,FP,CB]	[FB,CP,CB]
[PF,PF,PC]	0	0	-1/6	-1/6
[PF,PF,B]	0	1/6	-1/3	-1/6
[PF,PC,PC]	-1/6	0	0	1/6
[PF,PC,B]	-1/6	-1/6	1/6	1/6
[B,PF,PC]	-1/6	0	0	1/6
[B,PF,B]	1/6	-1/3	0	-1/2
[B,PC,PC]	1/6	-1/6	-1/6	-1/2
[B,PC,B]	0	-1/2	1/3	-1/6
	0	-1/3	1/6	-1/6

And the minimax optimal strategies are...

• A:

- If hold 1, then $\frac{5}{6}$ PassFold and $\frac{1}{6}$ Bet.
- If hold 2, then $\frac{1}{2}$ PassFold and $\frac{1}{2}$ PassCall.
- If hold 3, then $\frac{1}{2}$ PassCall and $\frac{1}{2}$ Bet.

Has both bluffing and underbidding...

• B:

- If hold 1, then $\frac{2}{3}$ FoldPass and $\frac{1}{3}$ FoldBet.
- If hold 2, then $\frac{2}{3}$ FoldPass and $\frac{1}{3}$ CallPass.
- If hold 3, then CallBet

Minimax value of game is $-\frac{1}{18}$ to A.

How to prove existence of NE

- Proof will be non-constructive.
- Notation:
 - Assume an $n \times n$ matrix.
 - Use $(p_1, \dots, p_n)$ to denote mixed strategy for row player, and $(q_1, \dots, q_n)$ to denote mixed strategy for column player.

Proof

- We'll start with Brouwer's fixed point theorem.
 - Let S be a bounded convex region in $\mathbb{R}^n$ and let $f:S \rightarrow S$ be a continuous function.
 - Then there must exist $x \in S$ such that $f(x)=x$.
 - x is called a "fixed point" of f .
- Simple case: S is the interval $[0,1]$.
- We will care about:
 - $S = \{(p,q): p,q \text{ are legal probability distributions on } 1,\dots,n\}$. I.e., $S = \text{simplex}_n \times \text{simplex}_n$

Proof (cont)

- $S = \{(p,q): p,q \text{ are mixed strategies}\}.$
- Want to define $f(p,q) = (p',q')$ such that:
 - f is continuous. This means that changing p or q a little bit shouldn't cause p' or q' to change a lot.
 - Any fixed point of f is a Nash Equilibrium.
- Then Brouwer will imply existence of NE.

Try #1

- What about $f(p,q) = (p',q')$ where p' is best response to q , and q' is best response to p ?
- Problem: not continuous:
 - E.g., penalty shot: If $p = (0.51, 0.49)$ then $q' = (1,0)$. If $p = (0.49, 0.51)$ then $q' = (0,1)$.

	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

Try #1

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$R =$

0	1
1	0

$C =$

0	-1
-1	0

Try #1

- What about $f(p,q) = (p',q')$ where p' is best response to q , and q' is best response to p ?
- Problem: also not necessarily well-defined:
 - E.g., if $p = (0.5,0.5)$ then q' could be anything.

R =

0	1
1	0

C =

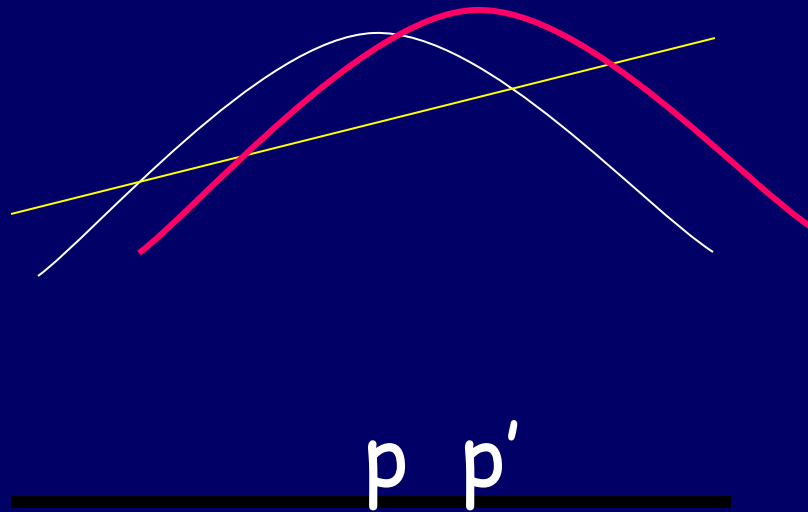
0	-1
-1	0

Instead we will use...

- $f(p,q) = (p',q')$ such that:
 - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
 - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]

$$p^T C q'$$

$$p'^T R q$$



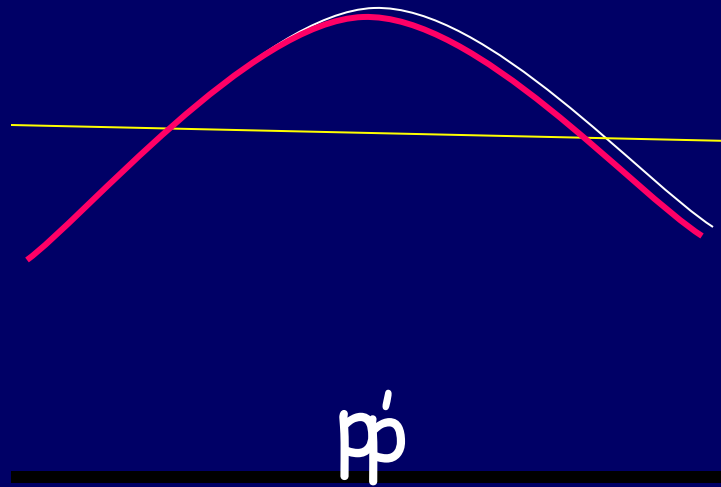
Note: quadratic + linear = quadratic.

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$$p^T C q'$$

$$p'^T R q$$



Note: quadratic + linear = quadratic.

Instead we will use...

- $f(p,q) = (p',q')$ such that:
 - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
 - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]
- f is well-defined and continuous since quadratic has unique maximum and small change to p,q only moves this a little.
- Also fixed point = NE. (even if tiny incentive to move, will move little bit).
- So, that's it!

Algorithmic Game Theory

Algorithmic issues in game theory:

- Computing equilibria / approximate equilibria in different kinds of games
- Understanding **quality** of equilibria in **load-balancing, network-design, routing, machine scheduling...**
- Analyzing **dynamics** of simple behaviors or adaptive (learning) algorithms: **quality guarantees, convergence,...**
- Design issues: constructing rules so that game will (ideally) have dominant-strategy equilibria with good properties.

End of Game Theory Intro